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# Affine Springer Fibers and Hitchin Fibers

## Set up

$X$ complete smooth curve $/\mathbb{k}$, or scheme of dimension 1, of finite type.

Langlands — geometric Langlands — Mirror symmetry

| | | | | |
|---|---|---|---|---|
| Global | $\mathbb{Z}$ | $X/\mathbb{F}_q$ | $X/\mathbb{C}$ | |
| Local | $\mathbb{Q}_p, \mathbb{R}, \mathbb{C}$ | $\mathbb{F}_q((t))$ | $\mathbb{C}((t))$ | $= F$ |

Locally compact
Harmonic analysis

Scholze
Fargues conjectures

$\longleftarrow$ Transport ideas

In the seminar we mostly worry about $\mathbb{C}((t))$.

Would like to understand the relationship between representations of (degenerate) $\boxed{\text{DAHA's}}$ and $\boxed{\text{affine Springer fibers.}}$

## Affine Springer fibers

Let $\mathfrak{g}$ be a reductive Lie algebra, $G$ corresponding

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group (adjoint)

$$\mathfrak{g} \xrightarrow{\text{char}} \mathfrak{g}/\!/G = \mathfrak{t}/W = \mathfrak{c}$$

$\mathfrak{t} \subset \mathfrak{g}$ max torus, $W$ Weyl group

Let $\gamma \in \mathfrak{g}(F)^{r.s.}$, $\mathcal{X}_\gamma = \{ g\, G(\mathcal{O}) \mid \operatorname{Ad}(g^{-1})\gamma \in \mathfrak{g}(\mathcal{O}) \}$ $\cap$ $G(F)/G(\mathcal{O})$

$\mathcal{O} = \mathcal{O}_F = \Bbbk[[t]] \subset F = \Bbbk((t))$.

For $G = GL_n$ we can think of $\mathcal{X}_\gamma$ as follows

$\mathcal{X}_\gamma = \{ \Lambda \subset F^n \mid \Lambda \text{ an } \mathcal{O}\text{-lattice } \gamma\Lambda \subset \Lambda \}$

<u>Ting</u> : $\mathcal{X}_\gamma \neq \emptyset$ iff $\operatorname{char}(\gamma) \in \mathfrak{c}(\mathcal{O})$ $[\, \mathfrak{c}^{r.s.}(\mathcal{O}) \,]$

Let's write $P = \operatorname{char}(\gamma) \in \mathcal{O}_F[x]$

$$\begin{array}{ccc} A = \mathcal{O}_F[x]/P(x) & \hookrightarrow & \mathfrak{g}(F) \\ \Psi & & \Psi \\ x & \longmapsto & \gamma \end{array}$$

| $\gamma$ r.s. $\Rightarrow$ $P$ also minimal polynomial |

Let $K = \operatorname{frac}(A)$, product of fields.

$K = F[x]/P(x) \cong F^n$

as a vector space

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Now $\mathcal{X}_\gamma \cong \{\text{Fractional ideal in } K\}$

Fractional ideal: f.g. $A$-submodule of $K$.

By Hensel's lemma happens precisely when char($\gamma$) is r.s.

Let $e \in F^n$ which we use as a base point
$F^n = K \cdot e$.

$$
\begin{array}{ccc}
\{\text{Fractional ideals}\} & \longleftrightarrow & \mathcal{X}_\gamma \\
\Psi & & \\
M & \longmapsto & Me \\
\{k \in K \mid ke \in \Lambda\} & \longleftarrow\!\shortmid & \Lambda
\end{array}
$$

If $P(x)$ factors into linear factors get $\mathbb{Z} \times \cdots \times \mathbb{Z}$, split semi-simple element.
$K^\times = \underbrace{F^\times \times \cdots \times F^\times}_{n}$

This describes the Springer fiber in terms of char($\gamma$).

## Hitchin fibers

The moduli space of Higgs bundles

$$\mathcal{M}_{G,\mathcal{L}} = \{ \mathcal{F} \text{ a principal } G\text{-bundle on } X,\ \varphi \in H^0(X, \mathrm{ad}(\mathcal{F}) \otimes \mathcal{L}) \}$$

Here $\mathcal{L}$ is line bundle on $X$. Best to think of $\mathcal{M}_{G,\mathcal{L}}$ as a stack. We use $\mathcal{L}$ to increase the "good part" of the stack, $\mathcal{L} \gg 0$

$$\mathcal{G}_{X,\mathcal{L}} = \underset{\substack{\uparrow \\ \text{constant } \mathcal{G} \text{ on } X}}{\mathcal{G}_X} \otimes_{\mathbb{G}_m} \mathcal{L}^* \qquad (\mathcal{L}^* = \mathcal{L} - \{0\})$$

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$\mathcal{A}_{G,\mathcal{L}} = H^0(X, \mathfrak{c}_{X,\mathcal{L}})$ and have the Hitchin map

$$\mathcal{M}_{G,\mathcal{L}} \xrightarrow{f} \mathcal{A}_{G,\mathcal{L}} \quad \text{given by}$$

$$\mathfrak{g} \longrightarrow \mathfrak{c}$$

Want to think of the Hitchin map as a characteristic polynomial.

$$\mathcal{A}^{r.s.}_{G,\mathcal{L}} = \{ a \in \mathcal{A}_{\mathcal{L},X} \text{ s.t. } a_F \in \mathfrak{c}^{r.s.}(F)/\mathbb{G}_m \}$$

Given $a \in \mathcal{A}^{r.s.}_{\mathcal{L},X}$ let's write $\mathcal{M}_a$ for the fiber of the Hitchin map. Want to describe this fiber.

Remark If $\mathcal{L} = \Omega_X$ then we are in the situation of the classical Hitchin fibration. In this case we have the fiber at 0 which is the global nilpotent cone

$$\mathcal{N}_X = \{ (\mathcal{F}, \varphi) \mid \varphi \text{ is nilpotent} \} = f^{-1}(0)$$

$$\cap$$

$$T^* \mathrm{Bun}_G = \mathcal{M}_{G,\Omega_X}$$

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$\mathcal{N}_X$ is Lagrangian and

$$\mathcal{N}_X = \bigcup T^*_{S_i} \mathrm{Bun}_G$$

some strata

Conj An irreducible perverse sheaf is a Hecke eigensheaf if and only if its characteristic variety is contained in $\mathcal{N}_X$.

Probably OK for $GL_n$ ~~and~~ in the sense that Hecke eigensheaves have this property. It should be true with multiplicities if think of $\mathcal{N}$ as a scheme theoretic fiber.

Note that it is a general principle that characters have nilpotent wave front set.

Back to regular semi-simple fibers and let us consider the case $G = GL_n$.

Let's spell out the spectral curve again.

Now $a \in H^0(X, \bigoplus_{i=0}^{n} \mathcal{L}^{\otimes i}) = \bigoplus H^0(X, \mathcal{L}^{\otimes i})\, x^{n-i}$

Can think of $f = \mathrm{char}$ as $(E \xrightarrow{\varphi} E \otimes \mathcal{L}) \mapsto \det(x - \varphi)$

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$$\Sigma = \mathrm{Tot}(\mathcal{L}) = \mathrm{Spec}\Big(\bigoplus_{i=0}^{\infty} \mathcal{L}^{\otimes -i}\Big), \quad \pi : \Sigma \longrightarrow X$$

$$\pi_* \mathcal{O}_\Sigma = \bigoplus_{i=0}^{\infty} \mathcal{L}^{(-i)} \qquad \mathcal{L}^{-n} \xrightarrow{\ a\ } \bigoplus_{i=0}^{\infty} \mathcal{L}^{\otimes(-i)}$$

$\downarrow$ adjunction

$$\longrightarrow \pi^* \mathcal{L}^{-n} \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathrm{Cok}$$

$$\begin{tikzcd}[column sep=small]
\boxed{C_a} \arrow[drr, "\pi"'] & = \mathrm{Spec}(\mathrm{Cok}) \subset & \Sigma \arrow[d, "\pi"] \\
 & & X
\end{tikzcd}$$

The Spectral Curve

Let's write $\overline{\mathrm{Pic}}(C_a) = \left\{ \begin{array}{c} \text{coherent torsion free sheaves on} \\ C_a \text{ of generic rank one} \end{array} \right\}$

$$\cup$$

$$\mathrm{Pic}(C_a) = \{\text{line bundles on } C_a\}$$

Fact $\boxed{\mathcal{M}_a \cong \overline{\mathrm{Pic}}(C_a)}$

Given $\mathcal{F} \in \overline{\mathrm{Pic}}(C_a)$ let's form $E = \pi_*(\mathcal{F})$ it is torsion free of rank $n$ on $X$ $\Rightarrow$ it is a vector bundle.
Now $\pi_*(\mathcal{O}_{C_a})$ acts on $E$, but $\mathcal{L}^{-1} \in \pi_*(\mathcal{O}_{C_a})$
$\Rightarrow$ $E \otimes \mathcal{L}^{-1} \longrightarrow E \iff E \longrightarrow E \otimes \mathcal{L}$.

## About Pic

Let's think about Pic for a moment.

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$$\mathrm{Pic}(C) = \prod_{x \in C} \mathcal{O}_x^* \backslash {\prod_{x\in C}}' F_x^* / F \quad , \quad F = k(C)$$

$$\mathrm{Pic}_x(C) = F_x^* / \mathcal{O}_x^*$$

$$\overline{\mathrm{Pic}}_x(C) = \{ \text{fractional ideals in } F_x^* \}$$

$$\boxed{\overline{\mathrm{Pic}}(C) \cong \mathrm{Pic}(C) \times^{\prod_{x \in C - C^{sm}} P_x(C)} \prod_{x \in C - C^{sm}} \overline{\mathrm{Pic}}_x(C)}$$

Express the fractional ideal in local terms.

Let us apply this to our situation.

We have $C_a \longrightarrow X$ and except for finitely many points on $X$ this is an étale map.

The point $a \in \mathcal{A}_{n,d} \rightsquigarrow a : X \to \mathfrak{c}/\mathbb{G}_m$ and outside a finite set of points $Z_a$ $a : X \longrightarrow \mathfrak{c}^{r.s.}$. At those points in $X - Z_a$ the map $C_a \longrightarrow X$ is étale.

Let $x \in X$ and form $a_x = a|\mathrm{Spec}(\mathcal{O}_x) \in \mathfrak{c}(\mathcal{O}_x) \cap \mathfrak{c}^{r.s.}(F_x)$

$\rightsquigarrow$ can lift to an $\underbrace{\text{elt}}_{\varepsilon(a_x)}$ in $\mathfrak{g}(F_x)^{r.s.}$ e.g. by taking the Kostant slice.

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Get an affine Springer fiber

$$\mathcal{X}_{\mathcal{E}(a_x)} = \mathcal{X}_{a_x} = \{ \text{Fractional ideals } K \}$$

$A = \mathcal{O}_{F_x}[u]/a_x(u)$ , $u$ the variable in the char polynomial

$K = \mathrm{Frac}(A)$

What is $\mathrm{Spec}(A)$ ?

$$\mathrm{Spec}(A) = C_{a,x}$$

$$\begin{tikzcd} C_{a,x} \arrow[r] \arrow[d] & C_a \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_x) \arrow[r] & X \end{tikzcd}$$

$$\boxed{\overline{\mathrm{Pic}}(C_a) = \mathrm{Pic}(C_a) \times^{\prod_{c \to Z} \mathrm{Pic}_c(C_a)} \prod_{c \to Z} \overline{\mathrm{Pic}}_c(C_a)}$$

But, now we see that $\boxed{\prod_{c \to x} \overline{\mathrm{Pic}}_c(C_a) = \mathcal{X}_{a_x}}$

The WEAVING theorem