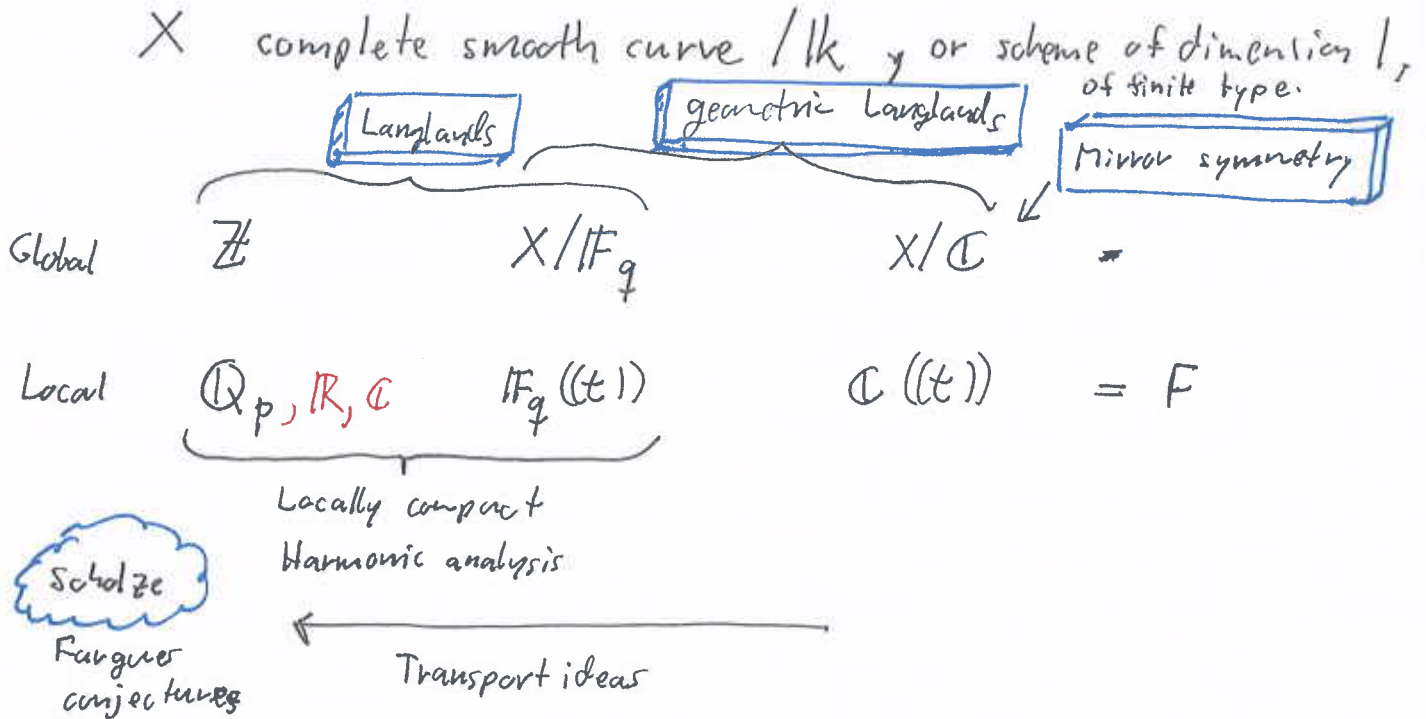


Affine Springer Fibers and Hitchin Fibers

Set up



In the seminar we mostly worry about  $\mathbb{C}((t))$ .

Would like to understand the relationship between representations of (degenerate) DAHA's and

affine Springer fibers.

Affine Springer fibers

Let  $\mathfrak{g}$  be a reductive Lie algebra,  $G$  corresponding

group (adjoint)

$$\mathfrak{g} \xrightarrow{\text{char}} \mathfrak{g}/G = \mathfrak{k}/W = \mathfrak{A}$$

$\mathfrak{k} \subset \mathfrak{g}$  max torus,  $W$  Weyl group

Let  $y \in \mathfrak{g}(F)$ ,  $\mathcal{K}_y = \{ g \in G(\mathcal{O}) \mid \text{Ad}(g^{-1})y \in \mathfrak{g}(\mathcal{O}) \}$  <sup>r.s.</sup>

$$\mathcal{O} = \mathcal{O}_F = \mathbb{k}[[t]] \subset F = \mathbb{k}((t)), \quad G(F)/G(\mathcal{O})$$

For  $G = GL_n$  we can think of  $\mathcal{K}_y$  as follows

$$\mathcal{K}_y = \{ \Lambda \subset F^n \mid \Lambda \text{ an } \mathcal{O}\text{-lattice } y\Lambda \subset \Lambda \}$$

Thing:  $\mathcal{K}_y \neq \emptyset$  iff  $\text{char}(y) \in G(\mathcal{O})$  [ $C^{\text{r.s.}}(\mathcal{O})$ ]

Let's write  $P = \text{char}(y) \in \mathcal{O}_F[x]$

$$A = \mathcal{O}_F[x]/P(x) \hookrightarrow \mathfrak{g}(F) \quad \left| \begin{array}{l} y^{\text{r.s.}} \Rightarrow P \text{ also} \\ \text{minimal polynomial} \end{array} \right.$$

$$\downarrow \quad \downarrow$$

$$x \longmapsto y$$

Let  $K = \text{frac}(A)$ , product of fields.

$$K = \mathbb{k}[x]/P(x) \cong F^n$$

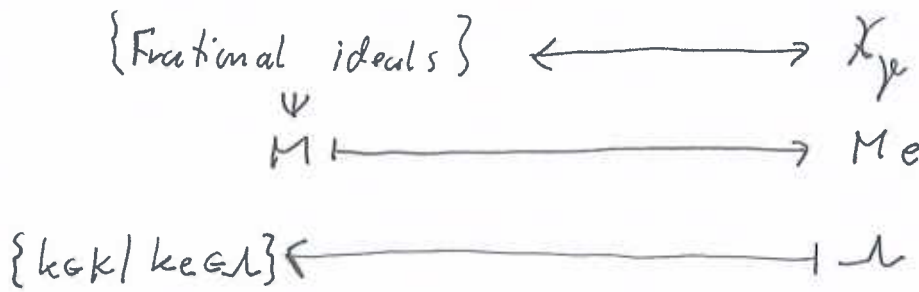
↑ as a vector space

Now  $X_{je} \cong \{ \text{Fractional ideal in } K \}$

Fractional ideal: f.g.  $A$ -submodule of  $K$ .

By Hensel's lemma happens precisely when  $\text{charge}(\phi)$  is v.s.

Let  $e \in F^n$  which we use as a base point  
 $F^n = K \cdot e$



↓

If  $P(x)$  factors into linear factors get  $\mathbb{Z} \times \dots \times \mathbb{Z}$ , split semi-simple element.  
 $K^* = \underbrace{F^* \times \dots \times F^*}_n$

This describes the Springer fiber in terms of  $\text{char}(\phi)$ .

Hitchin fibers

*The moduli space of Higgs bundles*

$$\mathcal{M}_{G, \mathcal{L}} = \{ \mathcal{F} \text{ a principal } G\text{-bundle on } X, \phi \in H^0(X, \text{ad}(\mathcal{F}) \otimes \mathcal{L}) \}$$

Here  $\mathcal{L}$  is line bundle on  $X$ . Best to think of  $\mathcal{M}_{G, \mathcal{L}}$  as a stack. We use  $\mathcal{L}$  to increase the "good part" of the stack,  $\mathcal{L} \gg 0$

$$\mathcal{G}_{X, \mathcal{L}} = \mathcal{G}_X \otimes_{\mathcal{G}_m} \mathcal{L}^* \quad (\mathcal{L}^* = \mathcal{L} - \{0\})$$

↑  
constant  $\mathcal{G}$  on  $X$

$\mathcal{A}_{G,\mathcal{L}} = H^0(X, \mathcal{E}_{X,\mathcal{L}})$  and have the

Hitchin map

$$\mathcal{M}_{G,\mathcal{L}} \xrightarrow{f} \mathcal{A}_{G,\mathcal{L}} \quad \text{given by}$$

$$g \longrightarrow a$$

Want to think of the Hitchin map as a characteristic polynomial

$$\mathcal{A}_{G,\mathcal{L}}^{\text{n.s.}} = \{ a \in \mathcal{A}_{\mathcal{L},X} \text{ s.t. } a_F \in \mathcal{E}^{\text{n.s.}}(F) / G_m \}$$

Given  $a \in \mathcal{A}_{\mathcal{L},X}^{\text{n.s.}}$  let's write  $\mathcal{M}_a$  for the fiber of the Hitchin map. Want to describe this fiber.

Remark If  $\mathcal{L} = \Omega_X$  then we are in the situation of the classical Hitchin fibration. In this case we have the fiber at 0 which is the global nilpotent cone

$$\mathcal{N}_X = \{ (\mathcal{E}, \varphi) \mid \varphi \text{ is nilpotent} \} = f^{-1}(0)$$

$$T^* \text{Bun}_G = \mathcal{M}_{G,\Omega_X}$$

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$\mathcal{N}_X$  is Lagrangian and

$$\mathcal{N}_X = \bigcup_{\substack{T_{S_i}^* \\ \uparrow \\ \text{some strata}}} \text{Bun}_G$$

Conj An irreducible perverse sheaf is a Hecke eigen sheaf if and only if its characteristic variety is contained in  $\mathcal{N}_X$ .

Probably OK for  $GL_n$  and in the sense that Hecke eigen sheaves have this property. It should be true with multiplicities if think of  $\mathcal{N}$  as a scheme theoretic fiber.

Note that it is a general principle that characters have nilpotent wave front set.

Back to regular semi-simple fibers and let us consider the case  $G = GL_n$ .

Let's spell out the spectral curve again.

$$\text{Now } \mathfrak{a} \in H^0(X, \bigoplus_{i=0}^n \mathcal{L}^{\otimes i}) = \bigoplus H^0(X, \mathcal{L}^{\otimes i}) \times \mathfrak{a}^{n-i}$$

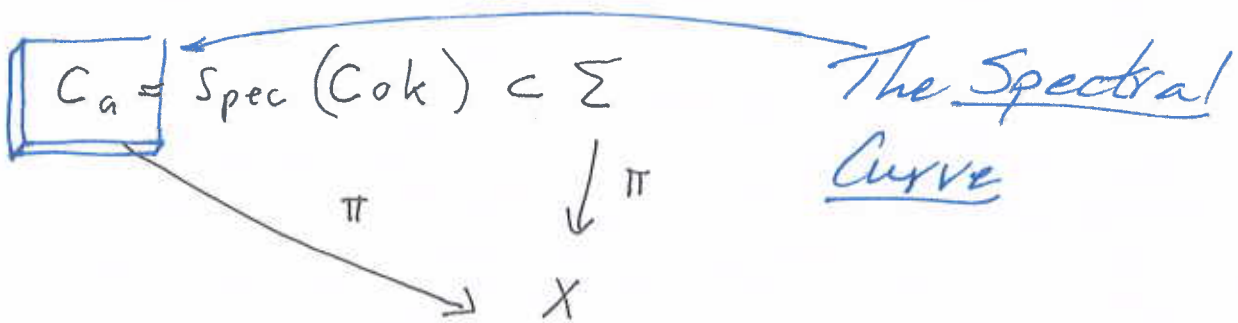
Can think of  $f = \text{char}$  as  $(E \xrightarrow{\varphi} E \otimes f) \mapsto \det(* - \varphi)$

$$\Sigma = \text{Tot}(d) = \text{Spec}\left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{\otimes -i}\right), \quad \pi: \Sigma \rightarrow X$$

$$\pi_* \mathcal{O}_{\Sigma} = \bigoplus_{i=0}^{\infty} \mathcal{L}^{\otimes -i} \quad \mathcal{L}^{-n} \xrightarrow{a} \bigoplus_{i=0}^{\infty} \mathcal{L}^{\otimes -i} \xrightarrow{\text{map}}$$

↓ adjunction

$$\rightarrow \pi^* \mathcal{L}^{-n} \rightarrow \mathcal{O}_{\Sigma} \rightarrow \text{Cok}$$



Let's write  $\overline{\text{Pic}}(C_a) = \left\{ \begin{array}{l} \text{coherent torsion free sheaves on } C_a \\ C_a \text{ of generic rank one} \end{array} \right\}$

$$\cup \text{Pic}(C_a) = \left\{ \text{line bundles on } C_a \right\}$$

Fact  $\mathcal{M}_a \cong \overline{\text{Pic}}(C_a)$

Given  $\mathcal{F} \in \overline{\text{Pic}}(C_a)$  let's form  $(\pi_*)^{-1}(\mathcal{F})$  it is torsion free of rank  $n$  on  $X \Rightarrow$  it is a vector bundle. Now  $\pi_* \mathcal{O}_{C_a}$  acts on  $E$ , but  $\mathcal{L}^{-1} \in \pi_* \mathcal{O}_{C_a} \Rightarrow E \otimes \mathcal{L}^{-1} \rightarrow E \Leftrightarrow E \rightarrow E \otimes \mathcal{L}$ .

### About Pic

Let's think about Pic for a moment.

$$\text{Pic}(C) = \prod_{x \in C} \mathcal{O}_x^* \backslash \prod_{x \in C} F_x^* / F, \quad F = k(C)$$

$$\text{Pic}_x(C) = F_x^* / \mathcal{O}_x^*$$

$$\overline{\text{Pic}}_x(C) = \{ \text{fractional ideals in } F_x^* \}$$

$$\overline{\text{Pic}}(C) \cong \text{Pic}(C) \times \prod_{x \in C - C^{\text{sm}}} \text{Pic}_x(C) \times \prod_{x \in C^{\text{sm}}} \overline{\text{Pic}}_x(C)$$

Express the fractional ideal in local terms.

Let us apply this to our situation.

We have  $C_a \longrightarrow X$  and except

for finitely many points on  $X$  this is an étale map.

The point  $a \in \mathbb{A}_{n,d} \rightsquigarrow a: X \rightarrow \mathbb{A}^1 / \mathbb{G}_m$   
and outside a finite set of points  $Z_a$   
 $a: X \rightarrow \mathbb{A}^1$ , At those points in  $X - Z_a$   
the map  $C_a \rightarrow X$  is étale.

Let  $x \in X$  and form  $a_x = a|_{\text{Spec}(\mathcal{O}_x)} \in \mathcal{Q}(\mathcal{O}_x) \cap C^{\text{vis}}(F_x)$   
 $\rightsquigarrow$  can lift to an elt in  $\mathcal{Q}(F_x)^{\text{v.r.}}$  e.g. by  
taking the Kottant slice.

Get an affine Springer fiber

$$\mathcal{X}_{\mathfrak{a}_x} = \mathcal{X}_{a_x} = \{ \text{Fractional ideals } K \}$$

$$A = \mathcal{O}_{F_x} \llbracket u \rrbracket / \mathfrak{a}_x(u), \quad u \text{ the variable in the char polynomial}$$

$$K = \text{Frac}(A)$$

What is  $\text{Spec}(A)$ ?

$$\parallel \\ C_{\mathfrak{a}, x}$$

$$\begin{array}{ccc} C_{\mathfrak{a}, x} & \rightarrow & C_{\mathfrak{a}} \\ \downarrow & & \downarrow \\ \text{Spec}(\mathcal{O}_x) & \rightarrow & X \end{array}$$

$$\overline{\text{Pic}}(C_{\mathfrak{a}}) = \text{Pic}(C_{\mathfrak{a}}) \times \prod_{c \rightarrow \mathbb{Z}} \text{Pic}_c(C_{\mathfrak{a}}) \quad \prod_{c \rightarrow \mathbb{Z}} \overline{\text{Pic}}_c(C_{\mathfrak{a}})$$

But, now we see that

$$\prod_{c \rightarrow x} \overline{\text{Pic}}_c(C_{\mathfrak{a}}) = \mathcal{X}_{a_x}$$

The WEAVING Theorem